TECHNICAL RESEARCH STATEMENT

EDWARD RICHMOND

1. INTRODUCTION

My research program explores relationships between the combinatorics and geometry of flag varieties, Schubert varieties and Coxeter groups. For over a century, Schubert varieties have been studied due to their rich combinatorial and geometric structures. Schubert calculus in relation to enumerative geometry is the main focus of Hilbert's 15th problem. Today, the study of Schubert geometry and combinatorics remains a very active field of mathematics and there are many open questions regarding Schubert varieties. Most famously, finding a combinatorially positive rule for calculating the Schubert polynomial structure constants remains open despite there existing a straightforward geometrical explanation for positivity.

My research falls into three categories:

- **Schubert varieties and fiber bundles:** This research program involves studying the combinatorial and geometric aspects of fiber bundle structures of Schubert varieties. One of the main applications of this work is the calculation of generation functions which enumerate smooth and rationally smooth Schubert varieties. Other applications include a pattern avoidance characterization of when Schubert varieties have complete parabolic bundle structures. More details on this research are given in Section 2.
- **Schubert Calculus and its applications:** This research program involves many different aspects of Schubert calculus. My work on this topic include studying Schubert calculus for Kac-Moody flag varieties, exploring saturation properties for T-equivariant cohomology, looking at recursive structures of the Belkale-Kumar product and finding applications of Schubert calculus to problems in frame theory and funcation analysis. More details on this research are given in Section 3.
- **Recent and future research:** Recently, I have been studying the Nash blow-ups of Schubert varieties in relation to Peterson translation along T-stable curves. I also have two projects that are currently in progress. The first is on understanding the combinatorics of the cohomology ring of Springer fibers and the second is on the product structure of noncommutative symmetric functions. More details on this work is given in Section 4.

2. SCHUBERT VARIETIES AND FIBER BUNDLES

Let G be a Lie group over an algebraically closed field and let W denote the Weyl group of G. The combinatorial properties of W are closely related to the geometry of Schubert varieties of the flag manifold G/B. For example, the Poincaré series of a Schubert variety $X(w)$ is the rank generating function of the Bruhat interval $[e, w]$. In [32], Slofstra and I developed a new insight into the relationship between the combinatorics of Weyl groups and the geometry of Schubert varieties through fiber bundle structures. Let P be parabolic subgroup of G. The projection map

$$\pi : G/B \to G/P$$

gives a P/B-fiber bundle structure on G/B. If W_P denotes the Weyl group P, then there is a unique parabolic decomposition of an element $w = vu$ where $u \in W_P$ and v is minimal length in the coset wW_P. Restricting the projection π to the Schubert variety $X(w)$ yields the projection

$$\pi : X(w) \to X^P(v)$$

where the generic fiber is isomorphic to $X(u)$. However π does not usually induce a fiber bundle structure on $X(w)$.

Question 2.1. *When is π restricted to $X(w)$ an $X(u)$-fiber bundle?*

Slofstra and I answer this question with the following combinatorial characterization.

Theorem 2.2. [32, Theorem 3.3] *The map π restricted to $X(w)$ a $X(u)$-fiber bundle if and only if u is maximal length in $[e, w] \cap W_P$.*

We remark that Theorem 2.2 holds for Schubert varieties of Kac-Moody flag varieties as well. We say a parabolic decomposition $w = vu$ is a *Billey-Postnikov (or BP)* decomposition if w satisfies either condition in Theorem 2.2. These decompositions were used by Billey and Postnikov in [11] to give a root subsystem characterization of smooth Schubert varieties in finite type. If $X(w)$ is a smooth or rationally smooth variety, then we have:

Theorem 2.3. [32, Theorem 3.6],[34, Theorem 1.1] *Let $X(w)$ be a Schubert variety of finite type or of affine type A. If $X(w)$ is (rationally) smooth, then w has a BP decomposition with respect to some maximal parabolic subgroup $P \subset W$.*

Moreover, if $X(w)$ is smooth, then the morphism $\pi : X(w) \to X^P(v)$ is smooth.

One immediate consequence of Theorem 2.3, is that a smooth Schubert variety in G/B is an iterated fiber bundle of smooth Schubert subvarieties of generalized Grassmannian flag manifolds (G/P where P is maximal). This fact was previously known only in type A [37, 40]. In [32, Theorem 3.8], we give a complete geometric description of smooth Schubert varieties in G/B by classifying all smooth Schubert varieties in generalized Grassmannians. Another consequence is that we prove the Billey-Crites conjecture in [10] which states that smooth Schubert varieties of affine type A correspond to affine permutations avoiding patterns 3412 and 4231.

Our interest in fiber bundle structures of Schubert varieties has its origins in earlier work where Slofstra and I study the combinatorics of Bruhat intervals $[e, w]$ where w is an element of some Coxeter group W. The property that a Schubert variety is smooth can be replaced with the combinatorial notion that the Bruhat interval $[e, w]$ is rank symmetric with respect to length. In other words, the Poincaré polynomial

$$P_w(t) := \sum_{x \in [e,w]} t^{\ell(x)}$$

is a palindromic polynomial. This condition is also equivalent to the Kazhdan-Lusztig polynomial indexed by (e, w) being equal to 1. In [31], Slofstra and I show that much of the theory on BP decompositions holds true for several families of Coxeter groups. For example, we prove the following analogue of Theorem 2.3:

Theorem 2.4. [31, Theorem 3.1] *Suppose W has no commuting Coxeter relations. If $P_w(t)$ is palindromic, then w has a BP-decomposition with respect to some proper maximal parabolic subgroup of W.*

We remark that Theorem 2.4 holds for right angled Coxeter groups as well. Theorem 2.4 allows us to construct a combinatorial "fiber bundle" structure on any Coxeter group element with a palindromic Poincaré polynomial. One surprising consequence is that the number of elements for which $P_w(t)$ is palindromic is finite for many infinitely large Coxeter groups [31, Corollary 3.5]. For uniform Coxeter groups, we calculate the generating function for the number of such elements in [31, Proposition 3.8].

2.1. **Enumerating smooth Schubert varieties.** In [33], Slofstra and I develop a model we call *Staircase diagrams* over a Dynkin graph which combinatorially encode the fiber bundle structures of a Schubert variety arising from Theorem 2.3. Our main application is that we derive the generating function for the number to smooth and rationally smooth Schubert varieties of any classical finite type. This generating function was previously only known in type A and was computed by Haiman [12, 20]. Specifically, define generating series

$$A(t) := \sum_{n=0}^{\infty} a_n\, t^n,\ B(t) := \sum_{n=0}^{\infty} b_n\, t^n,\ C(t) := \sum_{n=0}^{\infty} c_n\, t^n,\ D(t) := \sum_{n=3}^{\infty} d_n\, t^n,\ BC(t) := \sum_{n=0}^{\infty} bc_n\, t^n,$$

where the coefficients a_n, b_n, c_n, d_n denote the number of smooth Schubert varieties of types A_n, B_n, C_n and D_n respectively, and bc_n denotes the number of rationally smooth Schubert varieties of either type B_n or C_n.

Theorem 2.5. [33, Theorem 1.1] *Let $W(t) := \sum_n w_n\, t^n$ denote one of the above generating series, where $W = A$, B, C, D, or BC. Then*

$$W(t) = \frac{P_W(t) + Q_W(t)\sqrt{1-4t}}{(1-t)^2(1-6t+8t^2-4t^3)}$$

where $P_W(t)$ and $Q_W(t)$ are polynomials given in Table 1.

Type	$P_W(t)$	$Q_W(t)$
A	$(1-4t)(1-t)^3$	$t(1-t)^2$
B	$(1-5t+5t^2)(1-t)^3$	$(2t-t^2)(1-t)^3$
C	$1-7t+15t^2-11t^3-2t^4+5t^5$	$t-t^2-t^3+3t^4-t^5$
D	$(-4t+19t^2+8t^3-30t^4+16t^5)(1-t)^2$	$(4t-15t^2+11t^3-2t^5)(1-t)$
BC	$1-8t+23t^2-29t^3+14t^4$	$2t-6t^2+7t^3-2t^4$

TABLE 1. Polynomials in Theorem 2.5.

In [34, Theorem 1.2], Slofstra and I prove an analogous result for the generating function of smooth Schubert varieties of affine type A. One surprising consequence of these enumerations is that the asymptotic growth rate for the number of Schubert varieties is the same for each of the classical Lie types.

2.2. **Fiber bundle structures and pattern avoidance.** For Schubert varieties of finite type A, pattern avoidance have been used to characterize many geometery properties. Most notably, Lakishmbai and Sandhya's prove that a Schubert variety is smooth if and only if its corresponding permutation avoids the patterns 3412 and 4231. Since then, pattern avoidance has been used to characterize other properties such as being defined by inclusions [19], factorial [13] and a local complete intersection [38]. These results have recently been surveyed by Abe and Billey in [1]. In [2], Alland and I ask the following question:

Question 2.6. *Does the Coxeter theoretic condition for a fiber bundle in Theorem 2.2 have a pattern avoidance characterization in type A?*

We answer this question by developing a new notion of pattern avoidance called *split pattern avoidance*. Let $\mathrm{Fl}(n)$ denote the complete flag variety on $\mathbb{C}^n$ and $\mathrm{Gr}(r,n)$ denote the Grassmannian of r-dimensional subspaces of $\mathbb{C}^n$. There is a natural projection map

$$\pi_r : \mathrm{Fl}(n) \to \mathrm{Gr}(r,n)$$

given by projection onto the r-th factor $\pi(V_\bullet) := V_r$. In this case, Schubert varieties $X(w)$ in $\mathrm{Fl}(n)$ are indexed by permutations.

Theorem 2.7. [2, Theorem 1.1] *Let $w \in W$ be a permutation. The map π_r restricted to $X(w)$ is a fiber bundle if and only if w avoids the split patters $23|1$ and $3|12$ with respect to position r.*

One consequence is that we give a usual pattern avoidance characterization of Schubert varieties with complete parabolic bundle structures.

Theorem 2.8. [2, Theorem 1.3] *Let $w \in W$ be a permutation. Then $X(w)$ has a complete parabolic bundle structure if and only if w avoids the patterns 3412, 52341, 635241.*

3. SCHUBERT CALCULUS

The goal of Schubert calculus is understand the product structure of various cohomology theories of flag varieties and their generalizations with respect to a basis of Schubert classes. Questions can either be geometric or combinatorial in nature. In this section, I will discuss my research projects on Schubert calculus.

3.1. **Grassmannian Schubert calculus and applications.** This section is about two projects involving Schubert calculus of the Grassmannian $\mathrm{Gr}(r,n)$ of r-dimensional subspaces in $\mathbb{C}^n$. The cohomology ring $H^*(\mathrm{Gr}(r,n))$ has an additive basis of Schubert classes $\{\sigma_\lambda\}_{\lambda\in\Lambda}$, where Λ is the set of partitions whose Young diagrams are contained in an $r \times (n-r)$ rectangle. For any three partitions $\lambda, \mu, \nu \in \Lambda$ we can define the Littlewood-Richardson coefficients $c^\nu_{\lambda,\mu}$ by the product structure constants

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu\in\Lambda} c^\nu_{\lambda,\mu}\, \sigma_\nu.$$

The Littlewood-Richardson coefficients arise in several fields of mathematics including the representation theory of the general linear group, the combinatorics of symmetric functions, and quiver representations.

One remarkable application of Littlewood-Richardson coefficients is to the eigenvalue problem on sums of hermitian matrices. The following theorem is proved by the combined works of Klyachko [22] and Knutson and Tao [23].

Theorem 3.1. *([22, 23]) The coefficient $c^{\nu}_{\lambda,\mu} > 0$ if and only if there exist $r \times r$ hermitian matrices A, B, C with eigenvalues given by the partitions λ, μ, ν and*

$$A + B = C.$$

In joint work with Anderson and Yong [6], we are able to extend this result to the setting of torus-equivariant cohomology of the Grassmannian $H^*_T(\mathrm{Gr}(r, n))$. Define the structure constants $C^{\nu}_{\lambda,\mu}$ by the product of equivariant Schubert classes

$$\Sigma_\lambda \cdot \Sigma_\mu = \sum_{\nu \in \Lambda} C^{\nu}_{\lambda,\mu} \, \Sigma_\nu.$$

We have the following theorem (omitting some technical constraints).

Theorem 3.2. [6, Theorem 1.3] *The coefficient $C^{\nu}_{\lambda,\mu} > 0$ if and only if there exist $r \times r$ hermitian matrices A, B, C with eigenvalues given by the partitions λ, μ, ν and*

$$A + B \geq C.$$

Here a matrix $A \geq B$ if $A - B$ is positive semi-definite. Theorem 3.2 is proved by showing that Horn's inequalities, which determine when $c^{\nu}_{\lambda,\mu} > 0$, also determine when $C^{\nu}_{\lambda,\mu} > 0$ in the equivariant setting. As a corollary, we get an equivariant generalization of the celebrated saturation theorem.

Theorem 3.3. [6, Thoerem 1.1] *$C^{\nu}_{\lambda,\mu} > 0$ if and only if $C^{N\nu}_{N\lambda,N\mu} > 0$ for any $N > 0$.*

Another application of Theorem 3.1 is to frame theory, an important topic in functional analysis. Let $P_1, \ldots, P_k$ be a sequence of $N \times N$ orthogonal projection matrices and let $\mathbf{L} := (L_1, \ldots, L_k)$ denote the corresponding rank sequence (i.e. $\mathrm{rank}(P_i) = L_i$). We say that $P_1, \ldots, P_k$ is a *tight fusion frame* if there exists a real number α such that

$$\sum_{i=1}^{k} P_i = \alpha \mathbf{I}$$

where $\mathbf{I}$ denotes the identity matrix. Applications of fusion frames include sensor networks, coding theory, compressed sensing, and filter banks. In [14], together with Bownik and Luoto, we address the problem of classifying all $\mathbf{L}$ that are rank sequences of some tight fusion frame. Since orthogonal projection matrices are hermitian, we use Theorem 3.1 to prove the following classification.

Theorem 3.4. [14, Theorem 4.2] *$\mathbf{L} = (L_1, \ldots, L_k)$ is a tight fusion frame sequence if and only if*

$$\prod_{i=1}^{k} \sigma_{(N^{L_i})} \neq 0$$

in $H^*(\mathrm{Gr}(N, M + N))$ where $M := \sum_{i=1}^{k} L_i$ and the partition $(N^{L_i}) := \underbrace{(N, \ldots, N)}_{L_i}$.

This connection between frame theory and Schubert calculus yields many interesting results in both fields of mathematics. For example, using Schubert combinatorics, we produce new bounding estimates on tight fusion frames previously unknown in frame theory. Conversely, inspired by dualities found in frame theory, we construct new combinatorial identities for Littlewood-Richardson coefficients.

3.2. **Schubert calculus for Kac-Moody groups.** In joint work with Berenstein from [8], we study the Schubert calculus of the flag variety G/B corresponding to a Kac-Moody group G. The structure of G is encoded by a *generalized Cartan matrix* (GCM), defined to be a square matrix $A = (a_{i,j})$ where $a_{i,i} = 2$ and $a_{i,j} \in \mathbb{Z}_{<0}$ if $i \neq j$. Thus for each GCM, we can associate and study the cohomology ring $H^*(G/B)$.

Like the cohomology of the Grassmannian, $H^*(G/B)$ has an additive basis of Schubert classes indexed by W, the Weyl group of G. We define the structure constants $c_{u,v}^w$ by the product

$$\sigma_u \cdot \sigma_v = \sum_{w \in W} c_{u,v}^w \, \sigma_w.$$

In [8, Theorem 2.4], we give a formula for computing $c_{u,v}^w$ in terms of the GCM A. This formula is based on the work of Kostant and Kumar in [24] where they define and study nil-Hecke rings corresponding to Kac-Moody groups. While other formulas for Schubert structure constants exist (see [17]), it has been a long-standing open problem to find a formula that is "combinatorially positive". Although it is well known from the geometry of G/B that the Schubert structure constants are non-negative integers, there are no known combinatorial proofs of this positivity (except in a few very special cases). Our new formula satisfies the following property.

Theorem 3.5. [8, Theorem 2.16] *If the GCM $A = (a_{i,j})$ of G satisfies*

$$a_{i,j}a_{j,i} \geq 4 \tag{1}$$

for all i, j, then the formula for $c_{u,v}^w$ given in [8, Theorem 2.4] *is combinatorially positive.*

In other words, the formula we construct is completely algebraic and the proof of positivity does not rely on the geometry of G/B. The condition (1) is precisely the condition that the Weyl group W has no braid relations or commuting relations as a Coxeter group. Theorem 3.5 above and [8, Theorem 2.4] have both been extended to include Schubert structure constants for the torus-equivariant cohomology $H_T^*(G/B)$ in [8].

3.3. **Recursive formulas for structure constants.** Let $P \subseteq Q$ be a pair of parabolic subgroups of a complex Lie group G and consider the induced sequence of partial flag varieties

$$Q/P \hookrightarrow G/P \twoheadrightarrow G/Q.$$

When comparing the three flag varieties above, the variety G/P typically has the most complicated cohomology structure. In [29, 30], I develop a recursive formula to compute Schubert structure coefficients of $H^*(G/P)$ in terms of the simpler cohomology rings $H^*(Q/P)$ and $H^*(G/Q)$ under certain constraints.

Theorem 3.6. [30, Theorem 1.1] *Let $(w_1, w_2, w_3) \in (W^P)^3$ with parabolic decompositions $w_i = v_i u_i$ with respect to Q. If the triples (w_1, w_2, w_3) and (v_1, v_2, v_3) satisfy a certain numerical constraint, then*

$$c_{w_1,w_2}^{w_3} = c_{v_1,v_2}^{v_3} \cdot c_{u_1,u_2}^{u_3}.$$

One important class of coefficients satisfying these constraints of [30, Theorem 1.1] are coefficients $c_{u,v}^w$ corresponding to *Levi-movable* triples (u, v, w) defined by Belkale and Kumar [7]. In [27], Ressayre shows that the set of Levi-movable triples, with $c_{u,v}^w = 1$, indexes the interior faces of the eigencone corresponding to the group G. By applying the recursive formula [30, Theorem 1.1] to Ressayre's work, I am able to determine the inclusion relations of the faces of the eigencone.

In [28], Ressayre and I generalize the notion of Levi-movability to the setting of "branching Schubert calculus". Branching Schubert calculus refers to the problem of computing the comorphism on cohomology rings induced from an equivariant embedding of one flag variety into another. If we consider the diagonal embedding of a flag variety into two copies of itself, then the comorphism on cohomology is simply the cup product. Hence, branching Schubert calculus is a generalization of usual Schubert calculus. We use the generalized definition of Levi-movable to give a more elegant solution to the branching eigenvalue problem.

The main idea behind the proof of the recursive formula [30, Theorem 1.1] and its various applications to Levi-movability is to use the fact that Schubert structure coefficients count the number of points in the intersection of corresponding sets of Schubert varieties in general position. Since this intersection is transverse, we can apply tangent space analysis.

4. CURRENT AND FUTURE RESEARCH

The following are some project either recently completed or in progress.

4.1. **Nash blowups of Schubert varieties.** The Nash blow-up of a complex algebraic variety is the parameter space of tangent spaces over its smooth locus together with the limits of tangents spaces over its singular locus. One motivation for studying the Nash blow-up is that its tautological bundle serves as an analogue of the tangent bundle for singular varieties. The existence of such a blow-up has led to the development of a characteristic class theory for singular varieties [26]. For Schubert varieties, these classes have been extensively studied in [3, 4, 5, 21]. While the Nash blow-up is an extremely important object in class theory, its geometry and combinatorics is poorly understood. Recently in [35], Slofstra, Woo and I calculate the Nash blow-up of cominuscule Schubert varieties and show that the torus-fixed points of the Nash blow-up correspond to Peterson translates of the inversion set. This work is inspired by earlier work by Carrell and Kuttler in [15] where they define Peterson translation on T-stable varieties and use it to determine when a T-fixed point in the Schubert variety is smooth.

Theorem 4.1. [35, Theorem 2.1] *Let Δ, Δ_P denote the set of simple roots for G and P respectively and let $X(w)$ be a cominuscule Schubert variety in G/P. Further assume that w in minimal length in the coset wW_P.*

Then the Nash blow-up $X(w)$ is a Schubert variety. In particular, it is algebraically isomorphic to $\overline{BwQ}/Q$ for the standard parabolic subgroup $Q \subseteq P$, where Q is generated by the set of simple roots

$$\Delta_w := \{\beta \in \Delta_P \mid w(\beta) \in \Delta\}.$$

Theorem 4.1 has many consequences. First, it immediately implies that the Nash blow-up of $X(w)$ is a normal variety. Second, we use this result to give a new characterization of the smooth locus of $X(w)$. For Grassmannian Schubert varieties (which are all cominuscule), we determine when the Nash blow-up is a resolution of singularities. We also show that the Nash blow-up is a fiber product of left-peak and right-peak Zelevinsky resolutions.

4.2. **Cohomology of Springer fibers.** Precup and I are currently working on a project to study the geometry and topology of Springer fibers. Our goal is to define a T-equivariant analogue of the Garsia-Procesi (GP) basis for the cohomology of Springer fibers given in [18]. One application is that we want to explicitly compute the pull back of a Schubert class in cohomology. More precisely, let X_λ denote the Springer fiber corresponding to the partition λ and let

$$\phi : X_\lambda \to \mathrm{Fl}(n)$$

denote the inclusion map into the flag variety $\mathrm{Fl}(n)$.

Question 4.1. *Let σ_w denote a Schubert class in $H^*(\mathrm{Fl}(n))$. What is $\phi^*(\sigma_w)$ in terms of the GP basis of $H^*(X_\lambda)$?*

In [25], Kumar and Procesi develop a T-equivariant Borel model for the cohomology of Springer fibers which is naturally compatible with the classical Borel model on the flag variety. It is in this context that we hope to do our calculations by taking advantage of the additional torus structure on these varieties. Combinatorially, this reduces to calculating the structure constants of degenerate double Schubert polynomials with respect to our new T-equivariant GP-basis.

4.3. **Geometric Littlewood-Richardson rule.** There are geometric ways of computing LR coefficients $c^{\nu}_{\lambda,\mu}$ defined in Section 3.1. For any two partitions λ, μ, let $X(\lambda, \mu)$ denote the corresponding Richardson subvariety of the Grassmannian. By Kleiman's transversality, the cohomology class of the Richardson variety $[X(\lambda, \mu)] = \sigma_\lambda \cup \sigma_\mu$ and hence LR coefficients can be calculated by studying the geometry of $X(\lambda, \mu)$. In [39], Vakil gives a geometric Littlewood-Richardson rule in the form an algorithm derived by performing a series of flag degenerations which breaks the Richardson variety into a union of Schubert varieties. Combinatorially, Vakil's algorithm uses checker boards and can be diagramed by a rooted binary tree, where the root represents the Richardson variety and each node represents a component of the degeneration. Degeneration techniques have also been used by Coskun for Schubert calculus of the two step flag variety in [16]. Coskun's algorithm specializes to the Grassmannian and therefore computes LR coefficients as well. Coskun's algorithm is similar to Vakil's algorithm in that it can be diagramed by a rooted binary tree.

For any composition α, let $\mathbf{s}_\alpha$ denote the noncommutative Schur function as defined in [9]. The *noncommutative Littlewood-Richardson coefficients* $C^{\gamma}_{\alpha,\beta}$ are defined as the structure coefficients of the product

$$\mathbf{s}_\alpha \cdot \mathbf{s}_\beta = \sum_{\gamma} C^{\gamma}_{\alpha,\beta}\, \mathbf{s}_\gamma$$

in the algebra of noncommutative symmetric functions. Bessenrodt, Luoto and van-Willigenburg prove in [9] that the coefficients $C^{\gamma}_{\alpha,\beta}$ are nonnegative integers and are refinements of classical LR coefficients. More precisely, [9, Corollary 3.7] states for any compositions α, β with underlying partitions shapes $\lambda = \tilde{\alpha}$ and $\mu = \tilde{\beta}$, we have

$$c^{\nu}_{\lambda,\mu} = \sum_{\tilde{\gamma}=\lambda} C^{\gamma}_{\alpha,\beta}.$$

Tewari, van Willigenburg and I are working on a geometric explanation for this nonnegative refinement.

Problem 4.2. *Determine a geometric Littlewood-Richardson rule for noncommutative LR coefficients.*

With a slight modification to the geometric Littlewood-Richardson rule of Coskun in [16], we make the following conjecture.

Conjecture 4.3. *The geometric Littlewood-Richardson rule of Coskun* [16, Algorithm 3.24] *calculates the product* $\mathbf{s}_\alpha \cdot \mathbf{s}_\beta$ *in the case that* α *is a reverse partition and* β *is a partition.*

The output of Coskun's algorithm is a collection "shifted" Schubert varieties where classical LR coefficients are calculated by grouping those Schubert varieties in the collection with the same cohomology class. Conjecture 4.3 claims that the refinement of LR coefficients into noncommutative LR coefficients is captured by position data of each shifted Schubert variety. In other words, noncommutative LR coefficients are calculated by looking at position classes instead of cohomology classes.

We believe that Coskun's algorithm can be modified to compute other noncommutative LR coefficients.

Conjecture 4.4. *The geometric Littlewood-Richardson rule* [16, Algorithm 3.24] *generalizes to compute the product* $\mathbf{s}_\alpha \cdot \mathbf{s}_\beta$ *in the case that* α *is any composition and* β *is a partition.*

We also believe this noncommutative algorithm can be diagramed by a rooted binary tree similar to the algorithms in [16, 39]. A preliminary report of this work appears in [36].

References

[1] Hiraku Abe and Sara Billey. Consequences of the Lakshmibai-Sandhya theorem: the ubiquity of permutation patterns in Schubert calculus and related geometry. In *Schubert calculus—Osaka 2012*, volume 71 of *Adv. Stud. Pure Math.*, pages 1–52. Math. Soc. Japan, [Tokyo], 2016.

[2] Timothy Alland and Edward Richmond. Pattern avoidance and fiber bundle structures on Schubert varieties. *J. Combin. Theory Ser. A*, 154:533–550, 2018.

[3] Paolo Aluffi and Leonardo C. Mihalcea. Chern-Schwartz-MacPherson classes for Schubert cells in flag manifolds. *Compos. Math.*, 152(12):2603–2625, 2016.

[4] Paolo Aluffi, Leonardo C. Mihalcea, Joerg Schuermann, and Changjian Su. Shadows of characteristic cycles, verma modules, and positivity of chern-schwartz-macpherson classes of schubert cells. *preprint, arXiv:1709.08697*.

[5] Paolo Aluffi and Leonardo Constantin Mihalcea. Chern classes of Schubert cells and varieties. *J. Algebraic Geom.*, 18(1):63–100, 2009.

[6] David Anderson, Edward Richmond, and Alexander Yong. Eigenvalues of Hermitian matrices and equivariant cohomology of Grassmannians. *Compos. Math.*, 149(9):1569–1582, 2013.

[7] Prakash Belkale and Shrawan Kumar. Eigenvalue problem and a new product in cohomology of flag varieties. *Invent. Math.*, 166(1):185–228, 2006.

[8] Arkady Berenstein and Edward Richmond. Littlewood-Richardson coefficients for reflection groups. *Adv. Math.*, 284:54–111, 2015.

[9] C. Bessenrodt, K. Luoto, and S. van Willigenburg. Skew quasisymmetric Schur functions and noncommutative Schur functions. *Adv. Math.*, 226(5):4492–4532, 2011.

[10] Sara Billey and Andrew Crites. Pattern characterization of rationally smooth affine Schubert varieties of type *A*. *J. Algebra*, 361:107–133, 2012.

[11] Sara Billey and Alexander Postnikov. Smoothness of Schubert varieties via patterns in root subsystems. *Adv. in Appl. Math.*, 34(3):447–466, 2005.

[12] Miklós Bóna. The permutation classes equinumerous to the smooth class. *Electron. J. Combin.*, 5:Research Paper 31, 12 pp. (electronic), 1998.

[13] Mireille Bousquet-Mélou and Steve Butler. Forest-like permutations. *Ann. Comb.*, 11(3-4):335–354, 2007.

[14] Marcin Bownik, Kurt Luoto, and Edward Richmond. A combinatorial characterization of tight fusion frames. *Pacific J. Math.*, 275(2):257–294, 2015.

[15] James B. Carrell and Jochen Kuttler. Smooth points of T-stable varieties in G/B and the Peterson map. *Inventiones Mathematicae*, 151(2):353–379, 2003.

[16] Izzet Coskun. A Littlewood-Richardson rule for two-step flag varieties. *Invent. Math.*, 176(2):325–395, 2009.

[17] Haibao Duan. Multiplicative rule of Schubert classes. *Invent. Math.*, 159(2):407–436, 2005.

[18] A. M. Garsia and C. Procesi. On certain graded S_n-modules and the q-Kostka polynomials. *Adv. Math.*, 94(1):82–138, 1992.

[19] V. Gasharov and V. Reiner. Cohomology of smooth Schubert varieties in partial flag manifolds. *J. London Math. Soc. (2)*, 66(3):550–562, 2002.

[20] Mark Haiman. Enumeration of smooth schubert varieties. *preprint, 1992*.

[21] Benjamin F. Jones. Singular Chern classes of Schubert varieties via small resolution. *Int. Math. Res. Not. IMRN*, (8):1371–1416, 2010.

[22] Alexander A. Klyachko. Stable bundles, representation theory and Hermitian operators. *Selecta Math. (N.S.)*, 4(3):419–445, 1998.

[23] Allen Knutson and Terence Tao. The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture. *J. Amer. Math. Soc.*, 12(4):1055–1090, 1999.

[24] Bertram Kostant and Shrawan Kumar. The nil Hecke ring and cohomology of G/P for a Kac-Moody group G. *Adv. in Math.*, 62(3):187–237, 1986.

[25] Shrawan Kumar and Claudio Procesi. An algebro-geometric realization of equivariant cohomology of some Springer fibers. *J. Algebra*, 368:70–74, 2012.

[26] R. D. MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100:423–432, 1974.

[27] Nicolas Ressayre. Geometric invariant theory and the generalized eigenvalue problem. *Invent. Math.*, 180(2):389–441, 2010.

[28] Nicolas Ressayre and Edward Richmond. Branching Schubert calculus and the Belkale-Kumar product on cohomology. *Proc. Amer. Math. Soc.*, 139(3):835–848, 2011.

[29] Edward Richmond. A partial Horn recursion in the cohomology of flag varieties. *J. Algebraic Combin.*, 30(1):1–17, 2009.

[30] Edward Richmond. A multiplicative formula for structure constants in the cohomology of flag varieties. *Michigan Math. J.*, 61(1):3–17, 2012.

[31] Edward Richmond and William Slofstra. Rationally smooth elements of Coxeter groups and triangle group avoidance. *J. Algebraic Combin.*, 39(3):659–681, 2014.

[32] Edward Richmond and William Slofstra. Billey-Postnikov decompositions and the fibre bundle structure of Schubert varieties. *Math. Ann.*, 366(1-2):31–55, 2016.

[33] Edward Richmond and William Slofstra. Staircase diagrams and enumeration of smooth Schubert varieties. *J. Combin. Theory Ser. A*, 150:328–376, 2017.

[34] Edward Richmond and William Slofstra. Smooth Schubert varieties in the affine flag variety of type $\tilde{A}$. *European J. Combin.*, 71:125–138, 2018.

[35] Edward Richmond, William Slofstra, and Alexander Woo. The nash blow-up of a cominuscule schubert variety. *preprint, arXiv:1808.05918.*

[36] Edward Richmond, Vasu Tewari, and Stephanie van Willigenburg. A noncommutative geometric lr rule. *Proceedings of the 28th annual conference on Formal Power Series and Algebraic Combinatoircs*, DMTCS proc. BC:1039–1050, 2016.

[37] Kevin M. Ryan. On Schubert varieties in the flag manifold of $\mathrm{Sl}(n, \mathbb{C})$. *Mathematische Annalen*, 276(2):205–224, 1987.

[38] Henning Ulfarsson and Alexander Woo. Which Schubert varieties are local complete intersections? *Proc. Lond. Math. Soc. (3)*, 107(5):1004–1052, 2013.

[39] Ravi Vakil. A geometric Littlewood-Richardson rule. *Ann. of Math. (2)*, 164(2):371–421, 2006. Appendix A written with A. Knutson.

[40] James S Wolper. A combinatorial approach to the singularities of Schubert varieties. *Advances in Mathematics*, 76(2):184–193, 1989.